

Guaranteed Performance Leader-follower Control for Multi-agent Systems with Linear IQC-Constrained Coupling

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Abstract—This paper considers the leader-follower control problem for a linear multi-agent system with undirected topology and linear coupling subject to integral quadratic constraints (IQCs). A consensus-type control protocol is proposed based on each agent's states relative its neighbors. In addition a selected set of agents uses for control their states relative the leader. Using a coordinate transformation, the consensus analysis of the multi-agent system is recast as a decentralized robust control problem for an auxiliary interconnected large scale system. Based on this interconnected large scale system, sufficient conditions are obtained which guarantee that the system tracks the leader. These conditions guarantee a suboptimal bound on the system tracking performance. The effectiveness of the proposed method is demonstrated using a simulation example.

I. INTRODUCTION

Theoretical study of distributed coordination and control for multi-agent systems has received increasing attention in the past decade, due to its broad applications in unmanned air vehicles (UAVs), formation control, flocking, distributed sensor networks, etc [1]. As a result, much progress has been made in the study of cooperative control of multi-agent systems [2], [3], [4].

Efforts have recently been made to consider the leader-following consensus problem. It was noted in [5] that the control problem becomes much more complex if only a portion of the agents in the group has access to the leader. For example, the leader-following consensus problem for higher order multi-agent systems is presented for both fixed and switching topologies in [6]. In [7], distributed observers are designed for the system of second-order agents where an active leader to be followed moves with an unknown velocity, and the interaction topology has a switching nature. The consensus-based approach to observer-based synchronization of multiagent systems to the leader has further been explored in [8], [9].

The majority of leader following consensus-based control problems currently considered in the literature are focused on systems of independent agents. In many physical systems, however, interactions between agents are inevitable and must be taken into account. Examples of systems with a dynamical interaction between subsystems include power systems

and spacecraft control systems [10]. While in interacting systems decentralized control schemes can be used, in the examples mentioned above only relative states are available for measurement which poses an additional difficulty when using decentralized control. Hence, consensus based control strategies using relative state information have application in these problems; e.g., see [11].

In this paper, we are concerned with the leader-follower control problem for multi-agent systems coupled via linear unmodelled dynamics. Compared with the existing work in the field of the leader-follower consensus problem, interactions between the agents are regarded as uncertainty and are described in terms of time-domain integral quadratic constraints (IQCs) [16]. The IQC modelling is a well established technique to describe uncertain interactions between subsystems in a large scale system [12], [13], [17].

The IQC modelling allows us to analyze the effects of interactions between the agents from a robustness viewpoint. In this respect, the recent paper [22] is worth mentioning, which considers robust consensus protocols for synchronization of multiagent systems under additive uncertain perturbations with bounded H_∞ norm. Since the IQC conditions in our paper capture uncertain perturbations with bounded L_2 gain, we note a similarity between the two uncertainty classes. However, thanks to the time-domain IQC modelling, our paper goes beyond establishing a robust consensus. It develops the optimization approach to the leader-follower tracking problem, which provides a guarantee of performance of the leader-follower system under consideration (note that [22] consider a leaderless network). Namely, we pose the leader-follower tracking problem as a constrained optimization problem in which we optimize the worst-case consensus tracking performance of the system, as well as the cost associated with protocol actions. Our interest in robust performance guarantees is inspired by recent results on the distributed LQR design [21], [19].

The main contribution of the paper is a sufficient condition for the design of a guaranteed consensus tracking performance protocol for multi-agent systems subject to uncertain linear coupling. To derive such a condition, the underlying guaranteed performance leader following control problem is transformed into a guaranteed cost decentralized robust control for an auxiliary large scale system, which is comprised of coupled subsystems. The fact that the subsystems remain coupled after the transformation constitutes the main difference of our approach, compared with, e.g., [18], [19], where similar transformations resulted in a set of completely decoupled stabilization problems. Coupling

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between subsystems poses an additional difficulty compared with [18], [19], which stems from coupling between the agents. It is overcome using the minimax control design methodology of decentralized control synthesis [12], [13], [17].

The paper is organized as follows. Section II includes the problem formulation and some preliminaries. The main results are given in Section III. In section IV, the illustrative example is presented. Finally, the conclusions are given in Section V.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. Graph theory

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_N\}$ is a finite nonempty node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set of unordered pairs of nodes. The edge (i, j) in the edge set of an undirected graph means that nodes i and j obtain information from each other. Node i is called a neighbor of node j if $(i, j) \in \mathcal{E}$. The set of neighbors of node i is defined as $N_i = \{j | (i, j) \in \mathcal{E}\}$. \mathcal{G} is a simple graph if it has no self-loops or repeated edges. If there is a path between any two nodes of the graph \mathcal{G} , then the graph \mathcal{G} is connected, otherwise it is disconnected. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ of the undirected graph \mathcal{G} is defined as $a_{ij} = a_{ji} = 1$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = a_{ji} = 0$ otherwise. The degree matrix $\mathcal{D} = \text{diag}\{d_1, \dots, d_N\} \in \mathbb{R}^{N \times N}$ is a diagonal matrix, whose diagonal elements are $d_i = \sum_{j=1}^N a_{ij}$ for $i = 1, \dots, N$. The Laplacian matrix of the graph is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. It is symmetric when \mathcal{G} is undirected.

B. Problem Formulation

Consider a system consisting of N agents and a leader. The communication topology between the agents is described by a simple undirected connected graph \mathcal{G} . Dynamics of the i th agent are described by the equation

$$\dot{x}_i = Ax_i + B_1 u_i + B_2 \sum_{j \in N_i} \varphi(x_j(\cdot) \big|_0^t - x_i(\cdot) \big|_0^t), \quad (1)$$

where the notation $\varphi(y(\cdot) \big|_0^t)$ describes an operator mapping functions $y(s)$, $0 \leq s \leq t$, into \mathbb{R}^n . Also, $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^p$ is the control input. We note that the last term in (1) reflects a relative nature of interactions between agents.

Let $L_{2e}[0, \infty)$ be the space of functions $y(\cdot) : [0, \infty) \rightarrow \mathbb{R}^n$ such that $\int_0^T \|y(t)\|^2 dt < \infty$, $\forall T > 0$.

Assumption 1: φ is a linear operator mapping $L_{2e}[0, \infty) \rightarrow L_{2e}[0, \infty)$ which satisfies the following integral quadratic constraint (IQC) condition [16]. There exists a sequence $\{t_l\}$, $t_l \rightarrow \infty$, and a constant $d > 0$ such that for every t_l ,

$$\int_0^{t_l} \|\varphi(y(\cdot) \big|_0^t)\|^2 dt \leq \int_0^{t_l} \|y\|^2 dt + d, \quad \forall y \in L_{2e}[0, \infty). \quad (2)$$

The class of such operators will be denoted by Ξ_0 .

In addition to the system (1), suppose a leader is given. The dynamics of the leader, labeled 0, is expressed as

$$\dot{x}_0 = Ax_0, \quad (3)$$

where $x_0 \in \mathbb{R}^n$ is its state. We assume throughout the paper that the leader node can be observed from a subset of nodes of the graph \mathcal{G} . If the leader is observed by node i , we extend the graph \mathcal{G} by adding the edge $(0, i)$ with weighting gain $g_i = 1$, otherwise let $g_i = 0$. We refer to node i with $g_i \neq 0$ as a pinned or controlled node. Denote the pinning matrix as $G = \text{diag}[g_1, \dots, g_N] \in \mathbb{R}^{N \times N}$. The system is assumed to have at least one agent connected to the leader, hence $G \neq 0$.

Define synchronization error vectors as $e_i = x_0 - x_i$, $i = 1, 2, \dots, N$. Dynamics of these vectors satisfy the equation

$$\dot{e}_i = Ae_i - B_1 u_i - B_2 \sum_{j \in N_i} a_{ij} \varphi(e_i(\cdot) \big|_0^t - e_j(\cdot) \big|_0^t). \quad (4)$$

In this paper we are concerned with finding a control protocol for each node i of the form

$$u_i = -K \left\{ \sum_{j \in N_i} a_{ij} (x_j - x_i) + g_i (x_0 - x_i) \right\}, \quad (5)$$

where $K \in \mathbb{R}^{p \times n}$ is the feedback gain matrix to be found. As a measure of the system performance under this protocol, we will use the quadratic cost function (cf. [21]),

$$\mathcal{J}(u) = \sum_{i=1}^N \int_0^\infty \left(\frac{1}{2} \sum_{j \in N_i} (e_i - e_j)' Q (e_i - e_j) + g_i e_i' Q e_i + u_i' R u_i \right) dt, \quad (6)$$

where $Q = Q' > 0$ and $R = R' > 0$ are given weighting matrices, and u denotes the vector $u = [u_1' \dots u_N']'$. Each addend in the cost function (6) penalizes the i th system input, the disagreement between the i th and the j th system states, as well as the disagreement between the leader and the pinned agent which can observe the leader.

The problem in this paper is to find a control protocol (5) which solves the guaranteed performance leader following consensus control problem as follows:

Problem 1: Under Assumption 1, find a control protocol of the form (5) such that

$$\sup_{\Xi_0} \mathcal{J}(u) < \infty. \quad (7)$$

It will be shown later in the paper that (7) implies

$$\int_0^\infty \|e_i\|^2 dt < \infty \quad \forall i = 1, \dots, N. \quad (8)$$

Hence, solving Problem 1 will guarantee that all the agents synchronize to the leader in the L_2 sense.

C. Associated Guaranteed Cost Decentralized Control Problem

In this section, we introduce an auxiliary guaranteed cost decentralized control problem for an interconnected large scale system. Our approach follows [18], [19], however here it results in a collection of interconnected subsystems.

Taking the linearity of the operator φ into account, the closed loop large-scale system consisting of the collection of the error dynamics (4) and the protocols (5) can be written as

$$\dot{e} = (I_N \otimes A)e + ((\mathcal{L} + G) \otimes (B_1 K))e - ((\mathcal{L} + G) \otimes B_2)\Phi(t) + (G \otimes B_2)\Phi(t), \quad (9)$$

where

$$e = [e'_1 \ e'_2 \ \dots \ e'_N]',$$

$$\Phi(t) = [\varphi'(e_1(\cdot) |_0^t) \ \varphi'(e_2(\cdot) |_0^t) \ \dots \ \varphi'(e_N(\cdot) |_0^t)],$$

and \otimes denotes the Kronecker product.

It was shown in [14] that if the communication graph \mathcal{G} is connected and has at least one agent connected to the leader, then the symmetric matrix $\mathcal{L} + G$ is positive definite, Hence all its eigenvalues are positive.

Let $T \in \mathbb{R}^{N \times N}$ be an orthogonal matrix such that

$$T^{-1}(\mathcal{L} + G)T = J = \text{diag}[\lambda_1, \dots, \lambda_N]. \quad (10)$$

Also, let $\varepsilon = (T^{-1} \otimes I_n)e$, $\varepsilon = [\varepsilon'_1 \ \dots \ \varepsilon'_N]'$ and $\Psi(t) = (T^{-1} \otimes I_n)\Phi(t)$. Using this coordinate transformation, the system (9) can be represented in terms of ε , as

$$\dot{\varepsilon} = (I_N \otimes A + J \otimes (B_1 K))\varepsilon - (J \otimes B_2)\Psi(t) - ((T^{-1}GT) \otimes B_2)\Psi(t), \quad (11)$$

where

$$\Psi(t) = \begin{pmatrix} \varphi(\sum_{j=1}^N (T^{-1})_{1j} e_j(\cdot) |_0^t) \\ \vdots \\ \varphi(\sum_{j=1}^N (T^{-1})_{Nj} e_j(\cdot) |_0^t) \end{pmatrix} = \begin{pmatrix} \varphi(\varepsilon_1(\cdot) |_0^t) \\ \vdots \\ \varphi(\varepsilon_N(\cdot) |_0^t) \end{pmatrix}.$$

Here we used the assumption that $\varphi(\cdot)$ is a linear operator. It follows from (11) that

$$\dot{\varepsilon}_i = A\varepsilon_i + \lambda_i B_1 K \varepsilon_i + \sum_{j \neq i} \sum_p (T^{-1})_{ip} g_p T_{pj} B_2 \varphi(\varepsilon_j(\cdot) |_0^t) + (\sum_p (T^{-1})_{ip} g_p T_{pi} - \lambda_i) B_2 \varphi(\varepsilon_i(\cdot) |_0^t), \quad (12)$$

Then (11) can be regarded as a closed loop system consisting of N interconnected linear uncertain subsystems of the following form

$$\dot{\varepsilon}_i = A\varepsilon_i + B_{1i}\hat{u}_i + E_i\xi_i + L_i\eta_i, \quad (13)$$

each governed by a state feedback controller $\hat{u}_i = K\varepsilon_i$. Here we have used the following notation

$$\xi_i = \varphi(\varepsilon_i(\cdot) |_0^t), \quad (14)$$

$$\eta_i = [\xi'_1 \ \dots \ \xi'_{i-1} \ \xi'_{i+1} \ \dots \ \xi'_N]', \quad (15)$$

$$B_{1i} = \lambda_i B_1,$$

$$E_i = (f_{i,i} - \lambda_i)B_2,$$

$$L_i = B_2[f_{i,1}I, \dots, f_{i,(i-1)}I, f_{i,(i+1)}I, \dots, f_{i,N}I],$$

$$f_{i,j} = \sum_p T^{-1}_{ip} g_p T_{pj}.$$

According to Assumption 1, the linear coupling φ satisfies the integral quadratic constraints (2). This implies that the following two inequalities hold for all $i = 1, 2, \dots, N$:

$$\int_0^{t_i} \|\xi_i\|^2 dt \leq \int_0^{t_i} \|\varepsilon_i\|^2 dt + d, \quad (16)$$

$$\int_0^{t_i} \|\eta_i\|^2 dt \leq \int_0^{t_i} \sum_{j \neq i} \|\varepsilon_j\|^2 dt + (N-1)d. \quad (17)$$

It follows from (16) and (17) that the collection of uncertainty inputs $\xi_i, \eta_i, i = 1, 2, \dots, N$, represents an admissible local uncertainty and admissible interconnection inputs for the large-scale system (13); see [12], [13], [16], [17]. The sets of admissible uncertainty inputs and admissible interconnection inputs for the system (13) will be denoted by Ξ, Π , respectively. Then the above discussion can be summarized as follows: If φ satisfies the conditions in Assumption 1, then the corresponding signals (14), (15) belong to Ξ, Π , respectively.

Next, consider the performance cost (6). It is possible to show that

$$\mathcal{J}(u) = \int_0^\infty (e'((\mathcal{L} + G) \otimes Q)e + u'(I \otimes R)u) dt. \quad (18)$$

$$= \sum_{i=1}^N \int_0^\infty (\lambda_i \varepsilon'_i Q \varepsilon_i + \lambda_i^2 \varepsilon'_i K' R K \varepsilon_i) dt. \quad (19)$$

Thus we conclude that

$$\mathcal{J}(u) = \hat{\mathcal{J}}(\hat{u}), \quad (20)$$

where $\hat{u} = (\hat{u}'_1, \dots, \hat{u}'_N)'$, $\hat{u}_i = K\varepsilon_i$, and

$$\hat{\mathcal{J}}(\hat{u}) = \sum_{i=1}^N \int_0^\infty (\lambda_i \varepsilon'_i Q \varepsilon_i + \lambda_i^2 \hat{u}_i R \hat{u}_i) dt, \quad (21)$$

Now consider the auxiliary guaranteed cost decentralized control problem associated with the uncertain large scale system comprised of the subsystems (13), with uncertainty inputs (14) and interconnections (15), subject to the IQCs (16), (17). In this problem we wish to find a decentralized state feedback controller $\hat{u} = (\hat{u}'_1, \dots, \hat{u}'_N)'$, $\hat{u}_i = K\varepsilon_i$ such that

$$\sup_{\Xi, \Pi} \hat{\mathcal{J}}(\hat{u}) < \infty. \quad (22)$$

The discussion in this section can be summarized as follows.

Lemma 1: Under Assumption 1, if the state feedback controller $\hat{u} = (\hat{u}'_1, \dots, \hat{u}'_N)'$, $\hat{u}_i = K\varepsilon_i$, solves the auxiliary guaranteed cost decentralized control problem for the collection of systems (13) and the cost function (21), then the control protocol (5) with the gain matrix K solves Problem 1.

Proof: Since the state feedback controller $\hat{u}_i = K\varepsilon_i$ solves the auxiliary guaranteed cost decentralized control problem for the collection of the systems (13), then we have

$$\sup_{\Xi, \Pi} \hat{\mathcal{J}}(\hat{u}) < c. \quad (23)$$

Also we noted that every signal φ which satisfies Assumption 1 gives rise to an admissible uncertainty for the large-scale system consisting of subsystems (13). This implies that for any $\varphi \in \Xi_0$ with $u = (\mathcal{L} + G) \otimes Ke$, we have

$$\mathcal{J}(u) = \hat{\mathcal{J}}(\hat{u}) \leq \sup_{\Xi, \Pi} \hat{\mathcal{J}}(\hat{u}) < c. \quad (24)$$

It implies that the control protocol (5) with the same gain K solves Problem 1. ■

Note that since $\mathcal{L} + G$ is positive definite, then (8) follows from (20) and (18).

III. THE MAIN RESULTS

The main results of this paper are sufficient conditions under which the system (1), (5) tracks the leader with guaranteed consensus tracking performance.

Theorem 1: Let matrices $Y = Y' > 0, Y \in \mathbb{R}^{n \times n}$ and $F \in \mathbb{R}^{p \times n}$, and constants $\pi_i > 0, \theta_i > 0, i = 1, 2, \dots, N$, exist such that the following LMIs are satisfied simultaneously

$$\begin{bmatrix} Z_i & F' & YQ_i^{1/2} & Y & \mathbf{1}' \otimes Y \\ F & -\frac{1}{\lambda_i^2} R^{-1} & 0 & 0 & 0 \\ Q_i^{1/2} Y & 0 & -I & 0 & 0 \\ Y & 0 & 0 & -\frac{1}{\pi_i} I & 0 \\ \mathbf{1} \otimes Y & 0 & 0 & 0 & -\Theta_i^{-1} \end{bmatrix} < 0, \quad (25)$$

where

$$\begin{aligned} \mathbf{1} &= [1 \ \dots \ 1]' \in \mathbb{R}^{N-1}, \\ p_i &= f_{i,i} - \lambda_i, \\ q_i &= \left(\sum_{j \neq i} f_{i,j}^2 \right)^{1/2}, \\ Q_i &= \lambda_i Q, \end{aligned}$$

and

$$\begin{aligned} Z_i &= AY + YA' + \lambda_i F' B_1' + \lambda_i B_1 F \\ &+ (\pi_i^{-1} (p_i)^2 + \theta_i^{-1} q_i^2) B_2 B_2', \\ \Theta_i &= \text{diag}[\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_N] \otimes I_n. \end{aligned}$$

Then the control protocol (5) with $K = FY^{-1}$ solves Problem 1. Furthermore, this protocol guarantees the following performance bound

$$\sup_{\Xi_0} \mathcal{J}(u) \leq \sum_{i=1}^N (\pi_i + \theta_i(N-1))d + \sum_{i=1}^N e_i'(0)Y^{-1}e_i(0). \quad (26)$$

Proof: Using the Schur complement, the LMIs (25) can be transformed into the following Riccati inequality

$$\begin{aligned} AY + YA' + \left[\frac{p_i^2}{\pi_i} + \frac{q_i^2}{\theta_i} \right] B_2 B_2' + Y[Q_i + (\pi_i + \bar{\theta}_i)I]Y \\ + \lambda_i^2 F' R F + \lambda_i F' B_1' + \lambda_i B_1 F < 0, \end{aligned} \quad (27)$$

where $\bar{\theta}_i = \sum_{j \neq i} \theta_j$.

Consider the following Lyapunov function candidate for the interconnected systems (13), (16), (17):

$$V(\varepsilon) = \sum_{i=1}^N \varepsilon_i' Y^{-1} \varepsilon_i. \quad (28)$$

For the controller $\hat{u}_i = K\varepsilon_i$, we have

$$\begin{aligned} \frac{dV(\varepsilon)}{dt} &= 2 \sum_{i=1}^N \varepsilon_i' Y^{-1} p_i B_2 \xi_i + 2 \sum_{i=1}^N \varepsilon_i' Y^{-1} L_i \eta_i \\ &+ \sum_{i=1}^N \varepsilon_i' \left[Y^{-1} (A + \lambda_i B_1 K) + (A + \lambda_i B_1 K)' Y^{-1} \right] \varepsilon_i. \end{aligned} \quad (29)$$

Substitute $F = KY$ into the Riccati inequality (27). Then after pre- and post-multiplying (27) by Y^{-1} and substituting it into (29), we have

$$\begin{aligned} \frac{dV(\varepsilon)}{dt} &< - \sum_{i=1}^N \left(\varepsilon_i' \pi_i^{-1} p_i^2 Y^{-1} B_2 B_2' Y^{-1} \varepsilon_i \right. \\ &+ 2 \varepsilon_i' Y^{-1} p_i B_2 \xi_i - \pi_i \|\xi_i\|^2 + \pi_i \|\xi_i\|^2 - \pi_i \|\varepsilon_i\|^2 \\ &- \varepsilon_i' \theta_i^{-1} q_i^2 Y^{-1} B_2 B_2' Y^{-1} \varepsilon_i + 2 \varepsilon_i' Y^{-1} L_i \eta_i - \theta_i \|\eta_i\|^2 \\ &\left. + \theta_i \|\eta_i\|^2 - \bar{\theta}_i \|\varepsilon_i\|^2 \right) - \sum_{i=1}^N \varepsilon_i' \left[\lambda_i^2 K' R K + Q_i \right] \varepsilon_i. \end{aligned}$$

Using the following identity,

$$\sum_{i=1}^N \sum_{j \neq i} \theta_i \|\varepsilon_j\|_2^2 = \sum_{i=1}^N \bar{\theta}_i \|\varepsilon_i\|_2^2, \quad (30)$$

one has

$$\begin{aligned} \int_0^{t_l} \frac{dV(\varepsilon)}{dt} dt &< - \sum_{i=1}^N \int_0^{t_l} \varepsilon_i' (\lambda_i^2 K' R K + Q_i) \varepsilon_i dt \\ &+ \sum_{i=1}^N \pi_i \int_0^{t_l} (\|\xi_i\|^2 - \|\varepsilon_i\|^2) dt \\ &+ \sum_{i=1}^N \theta_i \int_0^{t_l} (\|\eta_i\|^2 - \sum_{j \neq i} \|\varepsilon_j\|^2) dt. \end{aligned}$$

Finally, using the IQCs (16) and (17) and $V(\varepsilon(t)) > 0$, we obtain

$$\begin{aligned} \sum_{i=1}^N \int_0^{t_l} \varepsilon_i' (\lambda_i^2 K' R K + Q_i) \varepsilon_i dt &< \sum_{i=1}^N (\pi_i + \theta_i(N-1))d \\ &+ V(\varepsilon(0)). \end{aligned} \quad (31)$$

The expression on the right hand side of the above inequality is independent of t_l . Letting $t_l \rightarrow \infty$ leads to

$$\mathcal{J}(\hat{u}) \leq \sum_{i=1}^N (\pi_i + \sum_{i=1}^N \theta_i(N-1))d + V(\varepsilon(0)). \quad (32)$$

Also (32) holds for arbitrary collection of inputs ξ_i, η_i that satisfy (16), (17), respectively. Then we conclude that

$$\sup_{\Xi, \Pi} \hat{\mathcal{J}}(\hat{u}) \leq \sum_{i=1}^N ((\pi_i + \sum_{i=1}^N \theta_i(N-1))d + e_i'(0)Y^{-1}e_i(0)). \quad (33)$$

The result of the theorem now follows from Lemma 1 and (24). ■

According to Theorem 1, one has to solve N coupled LMIs to obtain the control gain K . To simplify the calculation, we establish the following theorem which requires only one LMI to be feasible, as follows

$$\begin{bmatrix} \bar{Z} & Y(\bar{\lambda}Q)^{1/2} & Y & Y \\ (\bar{\lambda}Q)^{1/2}Y & -I & 0 & 0 \\ Y & 0 & -\frac{1}{\pi}I & 0 \\ Y & 0 & 0 & -\frac{1}{(N-1)\theta}I \end{bmatrix} < 0, \quad (34)$$

where

$$\begin{aligned} p^2 &= \max_i (p_i)^2, \\ q^2 &= \max_i (q_i^2), \\ \underline{\lambda} &= \min_i (\lambda_i), \\ \bar{\lambda} &= \max_i (\lambda_i), \end{aligned}$$

and

$$\bar{Z} = AY + YA' - \frac{\lambda^2}{\lambda^2} B_1 R^{-1} B_1' + \left[\frac{p^2}{\pi} + \frac{q^2}{\theta} \right] B_2 B_2'.$$

Lemma 2: Given $R = R' > 0$ and $Q > 0$, suppose the LMI (34) in variables $Y = Y' > 0$, $\pi^{-1} > 0$ and $\theta^{-1} > 0$ is feasible. Then the matrices and constants

$$Y, \quad F = -\frac{\lambda}{\lambda^2} R^{-1} B_1', \quad \pi_i = \pi, \quad \theta_i = \theta \quad (35)$$

are a feasible set of matrices and constants for the collection of LMIs (25).

Proof: Using the Schur complement, the above LMI (34) is equivalent to the following Riccati inequality

$$\begin{aligned} &AY + YA' - \frac{\lambda^2}{\lambda^2} B_1 R^{-1} B_1' + \left[\frac{p^2}{\pi} + \frac{q^2}{\theta} \right] B_2 B_2' \\ &+ Y[\bar{\lambda}Q + (\pi + \bar{\theta})I]Y < 0, \end{aligned} \quad (36)$$

where $\bar{\theta} = (N-1)\theta$.

Since $\underline{\lambda} \leq \lambda_i \leq \bar{\lambda}$, (36) implies that

$$\begin{aligned} &AY + YA' - \lambda_i \frac{\lambda}{\lambda^2} B_1 R^{-1} B_1' - \lambda_i \frac{\lambda}{\lambda^2} B_1 R^{-1} B_1' \\ &+ \frac{\lambda^2}{\lambda^2} B_1 R^{-1} B_1' + \left[\frac{p^2}{\pi} + \frac{q^2}{\theta} \right] B_2 B_2' \\ &+ Y[\lambda_i Q + (\pi + \bar{\theta})I]Y < 0. \end{aligned} \quad (37)$$

Substitute $F = -\frac{\lambda}{\lambda^2} R^{-1} B_1'$, $\pi_i = \pi$, $\theta_i = \theta$, $\bar{\theta}_i = (N-1)\theta = \sum_{j \neq i} \theta_j$, and let $p_i^2 = (f_{i,i} - \lambda_i)^2 \leq p^2$ and $q_i^2 = (\sum_{j \neq i} f_{i,j}^2) \leq q^2$, then we obtain

$$\begin{aligned} &AY + YA' + \lambda_i F' B_1' + \lambda_i B_1 F + \bar{\lambda}^2 F' R^{-1} F \\ &+ \left[\frac{p_i^2}{\pi_i} + \frac{q_i^2}{\theta_i} \right] B_2 B_2' + Y[\lambda_i Q + (\pi_i + \bar{\theta}_i)I]Y < 0. \end{aligned} \quad (38)$$

Noting that $\lambda_i^2 F' R^{-1} F \leq \bar{\lambda}^2 F' R^{-1} F$, we obtain the Riccati inequality (27) which is equivalent to (25). ■

Theorem 2: Given $R = R' > 0$ and $Q > 0$, suppose the LMI (34) in variables $Y = Y' > 0$, $\pi^{-1} > 0$ and $\theta^{-1} > 0$ is feasible. Then the control protocol (5) with $K = -\frac{\lambda}{\lambda^2} R^{-1} B_1' Y^{-1}$ solves Problem 1. Furthermore, this protocol guarantees the following performance bound

$$\sup_{\Xi_0} \mathcal{J}(u) \leq N(\pi + \theta(N-1))d + \sum_{i=1}^N e_i'(0)Y^{-1}e_i(0). \quad (39)$$

Proof: The proof readily follows from Lemma 2 and Theorem 1. ■

IV. EXAMPLE

To illustrate the proposed method, consider a system consisting of three identical pendulums coupled by two springs. Each pendulum is subject to an input as shown in Fig. 1. The dynamic of the coupled system is governed by the following equations

$$\begin{aligned} ml^2 \ddot{\alpha}_1 &= -ka^2(\alpha_1 - \alpha_2) - mgl\alpha_1 - u_1, \\ ml^2 \ddot{\alpha}_2 &= -ka^2(\alpha_2 - \alpha_3) - ka^2(\alpha_2 - \alpha_1) \\ &\quad - mgl\alpha_2 - u_2, \\ ml^2 \ddot{\alpha}_3 &= -ka^2(\alpha_3 - \alpha_2) - mgl\alpha_3 - u_3, \end{aligned} \quad (40)$$

where l is the length of the pendulum, a is the position of the spring, g is the gravitational acceleration constant, m is the mass of each pendulum, and k is the spring constant.

In addition to the three pendulums, consider the leader pendulum which is identical to those given. Its dynamics are described by the equation

$$ml^2 \ddot{\alpha}_0 = -mgl\alpha_0. \quad (41)$$

Choosing the state vectors as $x_0 = (\alpha_0, \dot{\alpha}_0)$, $x_1 = (\alpha_1, \dot{\alpha}_1)$, $x_2 = (\alpha_2, \dot{\alpha}_2)$ and $x_3 = (\alpha_3, \dot{\alpha}_3)$, the equation (40) and (41) can be written in the form of (1), (3), where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ -\frac{1}{ml^2} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ \frac{a^2}{ml^2} & 0 \end{bmatrix},$$

and

$$\varphi(x_j - x_i) = k(x_j - x_i).$$

The parameters of the coupled pendulum system are chosen as $m = 0.25kg$, $l = 1m$, $a = 0.5m$, $g = 10m/s^2$, $0 \leq k \leq 1N/m$.

The communication topology of the coupled system is an undirected graph, shown in Fig. 2. Suppose agent 1 is equipped with sensors which allow it to observe the leader.

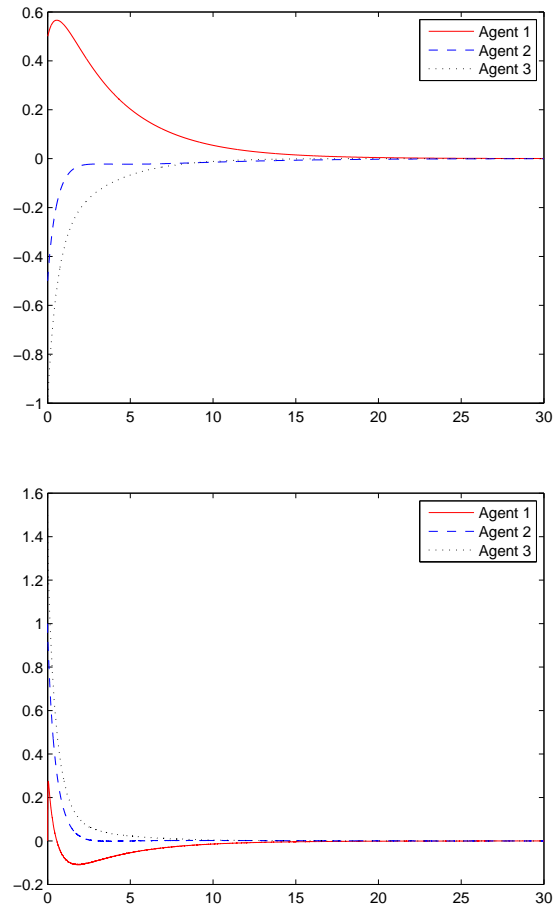


Fig. 4. Relative angles (the top figure) and relative velocities of the agents with respect to the leader, obtained using the algorithm based on Theorem 2.

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